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A new class of Banach spaces

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Abstract

In this paper, we construct a new class of separable Banach spaces $\mathbb{KS}^p$, for $1 \leqslant p \leqslant \infty$, each of which contains all of the standard L^p spaces, as well as the space of finitely additive measures, as compact dense embeddings. Equally important is the fact that these spaces contain all Henstock–Kurzweil integrable functions and, in particular, the Feynman kernel and the Dirac measure, as norm bounded elements. As a first application, we construct the elementary path integral in the manner originally intended by Feynman. We then suggest that $\mathbb{KS}^2$ is a more appropriate Hilbert space for quantum theory, in that it satisfies the requirements for the Feynman, Heisenberg and Schrödinger representations, while the conventional choice only satisfies the requirements for the Heisenberg and Schrödinger representations. As a second application, we show that the mixed topology on the space of bounded continuous functions, $\mathbf{C}_b[\mathbb{R}^n]$, used to define the weak generator for a semigroup $T(t)$, is stronger than the norm topology on $\mathbb{KS}^p$. (This means that, when extended to $\mathbb{KS}^p$, $T(t)$ is strongly continuous, so that the weak generator on $\mathbf{C}_b[\mathbb{R}^n]$ becomes a strong generator on $\mathbb{KS}^p$.)

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1. Introduction

The standard university analysis courses tend to produce a natural bias and unease concerning the use of finitely additive set functions as a basis for the general theory of integration (despite the efforts of Alexandrov [AX], Bochner [BR], Dubins [DU], Dunford and Schwartz [DS], de Finetti [DFN] and Yosida and Hewitt [YH]). (We should remember that the concept of measure was, and is, important for geometry and some, but not all, parts of analysis. In other parts, the concept of integral tends to dominate.)

Without denying an important place for countable additivity, Dubins ([DU], [BD], and [DUK]) argues forcefully for the intrinsic advantages in using finite additivity in the basic axioms of probability theory. (The penetrating analysis of the foundations of probability theory by de Finetti [DFN] also supports this position.) In a very interesting paper [DU], Dubins shows that the Wiener process has a number of 'cousins', related processes with the same finite dimensional distributions as the Wiener process. For example, there is one cousin with polynomial paths and another with piecewise linear paths. Since the Wiener measure is unique, these cousins must necessarily have finitely additive limiting distributions.

The most important of the finitely additive measures is that generated by the Henstock–Kurzweil integral (HK-integral), which generalizes the Lebesgue, Bochner and Pettis integrals. (It was discovered independently by Henstock [HS1] and Kurzweil [KW1].) The HK-integral is equivalent to the Denjoy integral. However, it is much easier to understand (and learn) compared to the Denjoy and Lebesgue integrals, and provides useful variants of the same theorems that have made the Lebesgue integral so important. Furthermore, it arises from a simple (transparent) generalization of the Riemann integral that is taught in elementary calculus. Loosely speaking, one uses a version of the Riemann integral with the interior points chosen first, while the size of the base rectangle around any interior point is determined by an arbitrary positive function defined at that point (see section 2). For more detail and different perspectives, see Gordon [GO], Henstock [HS], Kurzweil [KW] or Pfeffer [PF].

1.1. Purpose

Clearly, the most important factor preventing the widespread use of the HK-integral in engineering, mathematics and physics has been the lack of a natural Banach space structure for this class of functions (as is the case for the Lebesgue integral). The purpose of this paper is to introduce a new class of Banach spaces $\mathbb{KS}^p(\Omega)$, $1 \leqslant p \leqslant \infty$, with $\Omega \subset \mathbb{R}^n$, $(n = 1, 2, \ldots)$. These are all separable spaces that contain the corresponding $\mathbf{L}^p$ spaces as dense, continuous, compact embeddings. Our original interest was in the fact that each of these spaces contains the Denjoy integrable functions, as well as all the finitely additive measures. These spaces are perfect for the highly oscillatory functions that occur in quantum theory and nonlinear analysis.

1.2. Summary

In section 2, we give a brief introduction to the elementary HK-integral, its properties and relationship to the Lebesgue integral. In section 3, we construct the KS-spaces and derive some of the important properties of these spaces. We then prove that the Fourier transform and convolution operators have bounded extensions to $\mathbb{KS}^2$. These results are applied to the construction of the elementary path integral in the manner originally intended by Feynman. We then discuss our contention that $\mathbb{KS}^2$ is a more natural Hilbert space for quantum theory as compared to the conventional choice.

2. HK-integral

In this section, we introduce the HK-integral (in the simplest case) and present some of its properties. Our purpose is to provide those researchers unfamiliar with this integral a concrete sense of its simplicity. Proofs of all stated results can be found in Gordon [GO]. The general case can be found in Henstock [HS] or Pfeffer [PF].

Definition 1. *Let* $[a, b] \subset \mathbb{R}$*, let* $\delta(t)$ *map* $[a, b] \to (0, \infty)$*, and let* $\mathbf{P} = \{t_0, \tau_1, t_1, \tau_2, \ldots, \tau_n, t_n\}$*, where* $a = t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \cdots \leqslant \tau_n \leqslant t_n = b$*. We call* $\mathbf{P}$ *an HK-partition for* δ*, if for* $1 \leqslant i \leqslant n$*,* $t_{i-1}, t_i \in (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$*.*

Remark 2. Gordon writes $\mathbf{P} = \{(\tau_i, [t_{i-1}, t_i]) : 1 \leqslant i \leqslant n\}$ and calls $\{\tau_i\}$ the tags and $\{[t_{i-1}, t_i]\}$ the collection of tagged intervals. Also, the phrase nearly everywhere (n.e.) means except for a countable set.

Definition 3. *The function* $f(t)$*,* $t \in [a, b]$*, is said to have a HK-integral if there is a number* $F[a, b]$ *such that, for each* $\varepsilon > 0$*, there exists a function* δ *from* $[a, b] \to (0, \infty)$ *such that, whenever* $\mathbf{P}$ *is a HK-partition for* δ*, then (with* $\Delta t_i = t_i - t_{i-1}$*)*

$$\left| \sum_{i=1}^{n} \Delta t_i f(\tau_i) - F[a, b] \right| < \varepsilon.$$

In this case, we write $F[a, b] = (HK)\text{-}\int_a^b f(t)\, \mathrm{d}t$*.*

Theorem 4. *Let* $f(t) : [a, b] \to \mathbb{R}$*.*

(1) If $f(t)$ *is Lebesgue integrable on* $[a, b]$*, then it is HK-integrable on* $[a, b]$ *and HK-*$\int_a^b f(t)\, \mathrm{d}t = L\text{-}\int_a^b f(t)\, \mathrm{d}t$*.*
(2) If $f(t)$ *is HK-integrable and bounded on* $[a, b]$*, then it is Lebesgue integrable on* $[a, b]$*.*
(3) If $f(t)$ *is HK-integrable and nonnegative on* $[a, b]$*, then it is Lebesgue integrable on* $[a, b]$*.*
(4) If $f(t)$ *is HK-integrable on every measurable subset of* $[a, b]$*, then it is Lebesgue integrable on* $[a, b]$*.*

Theorem 5. *Let* $F : [a, b] \to \mathbb{R}$ *be continuous. If* F *is differentiable nearly everywhere on* $[a, b]$*, then* F' *is HK-integrable on* $[a, b]$ *and HK-*$\int_a^t F'(s)\, \mathrm{d}s = F(t) - F(a)$ *for each* $t \in [a, b]$*.*

The last result shows in what sense we can think of the HK-integral as the reverse of the derivative. (The result is not true for Lebesgue integrals. The standard example is $F'(t) = 2t \sin(\pi/t^2) - (2\pi/t)\cos(\pi/t^2)$ for all non-rational numbers on $0 < t < 1$ and equal to 0 at all rational points.)

3. $\mathbb{KS}^p$ spaces

In order to construct the spaces of interest, first recall that the HK-integral is equivalent to the Denjoy integral (see Henstock [HS] or Pfeffer [PF]). In the one-dimensional case, Alexiewicz [AL] has shown that the class $D(\mathbf{R})$, of Denjoy integrable functions, can be normed in the following manner: for $f \in D(\mathbf{R})$, define $\|f\|_D$ by

$$\|f\|_D = \sup_s \left| \int_{-\infty}^{s} f(r)\, \mathrm{d}r \right|.$$

It is clear that this is a norm, and it is known that $D(\mathbf{R})$ is not complete (see Alexiewicz [AL]). Replacing $\mathbf{R}$ by $\mathbf{R}^n$, for $f \in D(\mathbf{R}^n)$ we have the following generalization:

$$\|f\|_D = \sup_{r>0} \left| \int_{\mathbf{B}_r} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \right| = \sup_{r>0} \left| \int_{\mathbf{R}^n} \mathcal{E}_{\mathbf{B}_r}(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x} \right| < \infty, \tag{3.1}$$

where $\mathbf{B}_r$ is any closed cube of diagonal r centered at the origin in $\mathbf{R}^n$, with sides parallel to the coordinate axes, and $\mathcal{E}_{\mathbf{B}_r}(\mathbf{x})$ is the characteristic function of $\mathbf{B}_r$.

Now, fix n, and let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2 \ldots, x_n) \in \mathbf{R}^n\}$ such that x_i is rational for each i. Since this is a countable dense set in $\mathbf{R}^n$, we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}_i)$ be the closed cube centered at $\mathbf{x}_i$, with sides parallel to the coordinate axes and diagonal $r_l = 2^{-l}, l \in \mathbb{N}$. Now choose the natural lexicographical order which maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$, and let $\{\mathbf{B}_k, k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{\mathbf{B}_l(\mathbf{x}_i) \,|(l, i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$. Let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k$, so that $\mathcal{E}_k(\mathbf{x})$ is in $\mathbf{L}^p[\mathbf{R}^n] \cap \mathbf{L}^\infty[\mathbf{R}^n]$ for $1 \leqslant p < \infty$. Define $F_k(\cdot)$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$F_k(f) = \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{3.2}$$

It is clear that $F_k(\cdot)$ is a bounded linear functional on $\mathbf{L}^p[\mathbf{R}^n]$ for each k, $\|F_k\|_\infty \leqslant 1$ and, if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $\mathbf{L}^p[\mathbf{R}^n]$ for $1 \leqslant p \leqslant \infty$. Fix $t_k > 0$ such that $\sum_{k=1}^\infty t_k = 1$ and define a measure $\mathrm{d}\mathbf{P}(\mathbf{x}, \mathbf{y})$ on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$\mathrm{d}\mathbf{P}(\mathbf{x}, \mathbf{y}) = \left[\sum_{k=1}^\infty t_k \mathcal{E}_k(\mathbf{x})\mathcal{E}_k(\mathbf{y})\right] \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}.$$

We first construct our Hilbert space. Define an inner product $(\cdot)$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$\begin{aligned}(f, g) &= \int_{\mathbf{R}^n \times \mathbf{R}^n} f(\mathbf{x}) g(\mathbf{y})^* \mathrm{d}\mathbf{P}(\mathbf{x}, \mathbf{y}) \\ &= \sum_{k=1}^\infty t_k \left[\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x}\right] \left[\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{y}) g(\mathbf{y})\, \mathrm{d}\mathbf{y}\right]^*. \end{aligned} \tag{3.3}$$

We use a particular choice of t_k in Gill and Zachary [GZ], which is suggested by physical analysis in another context. We call the completion of $\mathbf{L}^1[\mathbf{R}^n]$, with the above inner product, the Kuelbs–Steadman space, $\mathbb{KS}^2[\mathbf{R}^n]$. Following suggestions of Gill and Zachary, Steadman [ST] constructed this space by adapting an approach developed by Kuelbs [KB] for other purposes. Her interest was in showing that $\mathbf{L}^1[\mathbf{R}^n]$ can be densely and continuously embedded in a Hilbert space which contains the HK-integrable functions. To see that this is the case, let $f \in D[\mathbf{R}^n]$, then

$$\|f\|_{\mathbf{KS}^2}^2 = \sum_{k=1}^\infty t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x}\right|^2 \leqslant \sup_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x}\right|^2 \leqslant \|f\|_D^2,$$

so $f \in \mathbb{KS}^2[\mathbf{R}^n]$.

Theorem 6. *For each p, $1 \leqslant p \leqslant \infty$, $\mathbb{KS}^2[\mathbf{R}^n] \supset \mathbf{L}^p[\mathbf{R}^n]$ as a dense subspace.*

Proof. By construction, $\mathbb{KS}^2[\mathbf{R}^n]$ contains $\mathbf{L}^1[\mathbf{R}^n]$ densely, so we need only show that $\mathbb{KS}^2[\mathbf{R}^n] \supset \mathbf{L}^q[\mathbf{R}^n]$ for $q \neq 1$. If $f \in \mathbf{L}^q[\mathbf{R}^n]$ and $q < \infty$, we have

$$\begin{aligned}\|f\|_{\mathbf{KS}^2} &= \left[\sum_{k=1}^\infty t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x}\right|^{\frac{2q}{q}}\right]^{1/2} \\ &\leqslant \left[\sum_{k=1}^\infty t_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x})\, |f(\mathbf{x})|^q\, \mathrm{d}\mathbf{x}\right)^{\frac{2}{q}}\right]^{1/2} \\ &\leqslant \sup_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) |f(\mathbf{x})|^q \mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leqslant \|f\|_q\,. \end{aligned}$$

Hence, $f \in \mathbb{KS}^2[\mathbf{R}^n]$. For $q = \infty$, first note that $\mathrm{vol}(\mathbf{B}_k)^2 \leqslant \left[\frac{1}{2\sqrt{n}}\right]^{2n}$, so we have

$$\|f\|_{\mathbf{KS}^2} = \left[\sum_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|^2\right]^{1/2}$$

$$\leqslant \left[\left[\sum_{k=1}^{\infty} t_k[\mathrm{vol}(\mathbf{B}_k)]^2\right][\mathrm{ess\,sup}\,|f|]^2\right]^{1/2} \leqslant \left[\frac{1}{2\sqrt{n}}\right]^n \|f\|_{\infty}.$$

Thus $f \in \mathbb{KS}^2[\mathbf{R}^n]$, and $\mathbf{L}^{\infty}[\mathbf{R}^n] \subset \mathbb{KS}^2[\mathbf{R}^n]$. □

The fact that $\mathbf{L}^{\infty}[\mathbf{R}^n] \subset \mathbb{KS}^2[\mathbf{R}^n]$, while $\mathbb{KS}^2[\mathbf{R}^n]$ is separable makes it clear in a very forceful manner that whether a space is separable or not depends on the topology. Before proceeding to additional study, we need to construct $\mathbb{KS}^p[\mathbf{R}^n]$.

To construct $\mathbb{KS}^p[\mathbf{R}^n]$ for all p and for $f \in \mathbf{L}^p$, define

$$\|f\|_{\mathbf{KS}^p} = \begin{cases} \left\{\sum_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|^p\right\}^{1/p}, & 1 \leqslant p < \infty, \\ \sup_{k \geqslant 1} \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|, & p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_{\mathbf{KS}^p}$ defines a norm on $\mathbf{L}^p$. If $\mathbf{KS}^p$ is the completion of $\mathbf{L}^p$ with respect to this norm, we have

Theorem 7. *For each q, $1 \leqslant q \leqslant \infty$, $\mathbb{KS}^p[\mathbf{R}^n] \supset \mathbf{L}^q[\mathbf{R}^n]$ as a dense continuous embedding.*

Proof. As in the previous theorem, by construction $\mathbb{KS}^p[\mathbf{R}^n]$ contains $\mathbf{L}^p[\mathbf{R}^n]$ densely, so we need only show that $\mathbb{KS}^p[\mathbf{R}^n] \supset \mathbf{L}^q[\mathbf{R}^n]$ for $q \neq p$. First, suppose that $p < \infty$. If $f \in \mathbf{L}^q[\mathbf{R}^n]$ and $q < \infty$, we have

$$\|f\|_{\mathbf{KS}^p} = \left[\sum_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|^{\frac{qp}{q}}\right]^{1/p}$$

$$\leqslant \left[\sum_{k=1}^{\infty} t_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x})\,|f(\mathbf{x})|^q\,\mathrm{d}\mathbf{x}\right)^{\frac{p}{q}}\right]^{1/p}$$

$$\leqslant \sup_k \left(\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x})\,|f(\mathbf{x})|^q\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leqslant \|f\|_q.$$

Hence, $f \in \mathbb{KS}^p[\mathbf{R}^n]$. For $q = \infty$, we have

$$\|f\|_{\mathbf{KS}^p} = \left[\sum_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|^p\right]^{1/p}$$

$$\leqslant \left[\left[\sum_{k=1}^{\infty} t_k[\mathrm{vol}(\mathbf{B}_k)]^p\right][\mathrm{ess\,sup}\,|f|]^p\right]^{1/p} \leqslant M\,\|f\|_{\infty}.$$

Thus $f \in \mathbb{KS}^p[\mathbf{R}^n]$, and $\mathbf{L}^{\infty}[\mathbf{R}^n] \subset \mathbb{KS}^p[\mathbf{R}^n]$. The case $p = \infty$ is obvious. □

Theorem 8. *For $\mathbb{KS}^p$, $1 \leqslant p \leqslant \infty$, we have*

(1) If $f, g \in \mathbb{KS}^p$, then $\|f + g\|_{\mathbf{KS^p}} \leqslant \|f\|_{\mathbf{KS^p}} + \|g\|_{\mathbf{KS^p}}$ (Minkowski inequality).
(2) If K is a weakly compact subset of $\mathbf{L}^p$, it is a compact subset of $\mathbb{KS}^p$.
(3) If $1 < p < \infty$, then $\mathbb{KS}^p$ is uniformly convex.
(4) If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then the dual space of $\mathbb{KS}^p$ is $\mathbb{KS}^q$.
(5) $\mathbb{KS}^{\infty} \subset \mathbb{KS}^p$, for $1 \leqslant p < \infty$.

Proof. The proof of (1) follows from the classical case for sums. The proof of (2) follows from the fact that, if $\{f_n\}$ is any weakly convergent sequence in K with limit f, then

$$\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})[f_n(\mathbf{x}) - f(\mathbf{x})]\,\mathrm{d}\mathbf{x} \to 0$$

for each k. It follows that $\{f_n\}$ converges strongly to f in $\mathbb{KS}^p$.

The proof of (3) follows from a modification of the proof of the Clarkson inequalities for l^p norms.

In order to prove (4), observe that, for $p \neq 2$, $1 < p < \infty$, the linear functional

$$L_g(f) = \|g\|_{\mathbf{KS}^p}^{2-p} \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) g(\mathbf{x})\,\mathrm{d}\mathbf{x} \right|^{p-2} \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y}) f(\mathbf{y})^*\,\mathrm{d}\mathbf{y}$$

is a unique duality map on $\mathbb{KS}^q$ for each $g \in \mathbb{KS}^p$ and that $\mathbb{KS}^p$ is reflexive from (3). To prove (5), note that $f \in \mathbb{KS}^\infty$ implies that $\left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|$ is uniformly bounded for all k. It follows that $\left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|^p$ is uniformly bounded for each p, $1 \leqslant p < \infty$. It is now clear from the definition of $\mathbb{KS}^\infty$ that

$$\left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right|^p \right]^{1/p} \leqslant \|f\|_{\mathbb{KS}^\infty} < \infty.$$

Note that, since $\mathbf{L}^1[\mathbf{R}^n] \subset \mathbb{KS}^p[\mathbf{R}^n]$ and $\mathbb{KS}^p[\mathbf{R}^n]$ is reflexive for $1 < p < \infty$, we see that the second dual $\left\{\mathbf{L}^1[\mathbf{R}^n]\right\}^{**} = \mathfrak{M}[\mathbf{R}^n] \subset \mathbb{KS}^p[\mathbf{R}^n]$. Recall that $\mathfrak{M}[\mathbf{R}^n]$ is the space of bounded finitely additive set functions defined on the Borel sets $\mathfrak{B}[\mathbf{R}^n]$. This space contains the Dirac delta measure and the free-particle Green's function for the Feynman integral. We will return to $\mathfrak{M}[\mathbf{R}^n]$ in the next section.

In many applications, it is convenient to formulate problems on one of the standard Sobolev spaces $\mathbf{W}_p^m(\mathbb{R}^n)$. We close this section with an admittedly incomplete, but most direct approach to the corresponding extension for the $\mathbb{KS}^p$ spaces. First recall that the space

$$\mathbf{X}_p^m(\mathbb{R}^n) = \{B_\alpha * g = (I - \triangle)^{-\alpha/2} g : g \in \mathbf{L}^p(\mathbb{R}^n), 0 < \alpha < n, 0 < \alpha < m\}$$

coincides with $\mathbf{W}_p^m(\mathbb{R}^n)$ when $1 < p < \infty$ and $m > 0$, where B_α is the Bessel potential of order α, $\triangle$ is the Laplacian and $*$ is the convolution operator.

Theorem 9. *The completion of* $\mathbf{X}_p^m(\mathbb{R}^n)$ *relative to the* $\mathbb{KS}^p(\mathbb{R}^n)$ *norm defines the space* $\mathbb{KS}_p^m(\mathbb{R}^n)$*, which contains* $\mathbf{W}_p^m(\mathbb{R}^n)$ *as a continuous dense and compact embedding.*

Proof. Since $B_\alpha \in \mathbf{L}^1(\mathbb{R}^n)$, we see from Young's inequality for convolutions that $B_\alpha * g \in \mathbf{L}^p(\mathbb{R}^n)$ if $g \in \mathbf{L}^p(\mathbb{R}^n)$ and $0 < \alpha < n$. □

In closing, we first recall that a function f such that $\int_K |f(\mathbf{x})|^p\,\mathrm{d}\mathbf{x} < \infty$ for every compact set K in $\mathbb{R}^n$ is said to be in $\mathbf{L}_{\mathrm{loc}}^p(\mathbb{R}^n)$. We can easily show that $\mathbf{L}_{\mathrm{loc}}^p(\mathbb{R}^n) \subset \mathbb{KS}^q(\mathbb{R}^n)$, $1 \leqslant q \leqslant \infty$, for all p, $1 \leqslant p \leqslant \infty$. This means that $\mathbb{KS}^q(\mathbb{R}^n)$ contains a large class of distributions (see Adams [A]).

3.1. Extension of Fourier and convolution operators

Let $L[\mathcal{B}]$, $L[\mathcal{H}]$ denote the bounded linear operators on $\mathcal{B}$, $\mathcal{H}$ respectively, where we assume that the separable Banach space $\mathcal{B}$ is a continuous dense embedding in the separable Hilbert space $\mathcal{H}$. The following is the major result in Gill *et al* [GBZS]. It generalizes the well-known result of von Neumann [VN1] for bounded operators on Hilbert spaces.

Theorem 10. *Let $\mathcal{B}$ be a separable Banach space and let A be a bounded linear operator on $\mathcal{B}$. Then A has a well-defined adjoint A^* defined on $\mathcal{B}$ such that*

(1) the operator $A^*A \geqslant 0$ (maximal accretive),
(2) $(A^*A)^* = A^*A$ (selfadjoint), and
(3) $I + A^*A$ has a bounded inverse.

The proof depends on the fact that, given a separable Banach space $\mathcal{B}$, there always exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ as continuous dense embeddings, with $\mathcal{H}_1$ determined by $\mathcal{H}_2$ (see [GBZS]). If A is any bounded linear operator on $\mathcal{B}$, we define A^* by

$$A^*x = J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(x), \tag{3.4}$$

where A_1 is A restricted to $\mathcal{H}_1$, $J_2|_{\mathcal{B}}$ maps $\mathcal{B}$ into $\mathcal{H}'_2$ and J_1^{-1} maps $\mathcal{H}'_1$ onto $\mathcal{H}_1$.

It is not clear that A need have a bounded extension to $\mathcal{H}_2$. On the other hand, the theorem by Lax [LX] states that

Theorem 11. *If A is a bounded linear operator on $\mathcal{B}$ such that A is selfadjoint (i.e., $(Ax, y)_{\mathcal{H}_2} = (x, Ay)_{\mathcal{H}_2}$ for all $x, y, \in \mathcal{B}$), then A is bounded on $\mathcal{H}_2$ and $\|A\|_{\mathcal{H}_2} \leqslant k\,\|A\|_{\mathcal{B}}$ with k constant.*

Since A^*A is selfadjoint on $\mathcal{B}$, it is natural to expect that the same is true on $\mathcal{H}_2$. However, this need not be the case. To obtain a simple counterexample, recall that, in standard notation, the simplest class of bounded linear operators on $\mathcal{B}$ is $\mathcal{B} \otimes \mathcal{B}'$, in the sense that

$$\mathcal{B} \otimes \mathcal{B}' : \mathcal{B} \to \mathcal{B}, \qquad \text{by} \quad Ax = (b \otimes b')x = \langle x, b'\rangle b.$$

Thus, if b' is in $\mathcal{B}'\backslash\mathcal{H}_2'$, then $J_2\{J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(x)\}$ need not be in $\mathcal{H}_2'$, so that A^*A is not defined as an operator on all of $\mathcal{H}_2$ and thus, cannot have a bounded extension. We can now state the correct extension of theorem 11.

Theorem 12. *Let A be a bounded linear operator on $\mathcal{B}$. If $\mathcal{B}' \subset \mathcal{H}_2$, then A has a bounded extension to $L[\mathcal{H}_2]$, with $\|A\|_{\mathcal{H}_2} \leqslant k\,\|A\|_{\mathcal{B}}$ with k constant.*

Proof. The proof is now easy if, we observe that, with the stated condition, $J_2\left\{J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(x)\right\}$ is in $\mathcal{H}_2'$ for all $x \in \mathcal{B}$. It follows that, for any bounded linear operator A defined on $\mathcal{B}$, the operator $T = A^*A$ is selfadjoint on $\mathcal{H}_2$. Thus, by Lax's theorem, T is bounded on $\mathcal{H}_2$, with $\|A^*A\|_{\mathcal{H}_2} = \|A\|_{\mathcal{H}_2}^2 \leqslant \|A^*A\|_{\mathcal{B}} \leqslant k\|A\|_{\mathcal{B}}^2$, where $k = \inf\left\{M | \|A^*A\|_{\mathcal{B}} \leqslant M\|A\|_{\mathcal{B}}^2\right\}$. □

We can now use theorem 12 to prove that $\mathfrak{F}$ and $\mathfrak{C}$, the Fourier (transform) operator and the convolution operator respectively, defined on $\mathbf{L}^1[\mathbf{R}^n]$, have bounded extensions to $\mathbb{KS}^2[\mathbf{R}^n]$. It should be noted that this theorem also implies that both operators have bounded extensions to all $\mathbf{L}^p[\mathbf{R}^n]$ spaces for $1 \leqslant p < \infty$. This is the first proof based on functional analysis, while the traditional proof is obtained via rather deep methods of (advanced) real analysis.

Theorem 13. *Both $\mathfrak{F}$ and $\mathfrak{C}$ extend to bounded linear operators on $\mathbb{KS}^2[\mathbf{R}^n]$.*

Proof. To prove our result, first note that $C_0[\mathbf{R}^n]$, the bounded continuous functions on $\mathbf{R}^n$ which vanish at infinity, is contained in $\mathbb{KS}^2[\mathbf{R}^n]$. Now $\mathfrak{F}$ is a bounded linear operator from $\mathbf{L}^1[\mathbf{R}^n]$ to $C_0[\mathbf{R}^n]$, so we can consider it as a bounded linear operator from $\mathbf{L}^1[\mathbf{R}^n]$ to $\mathbb{KS}^2[\mathbf{R}^n]$. Since $\mathbf{L}^1[\mathbf{R}^n]$ is dense in $\mathbb{KS}^2[\mathbf{R}^n]$ and $\mathbf{L}^\infty[\mathbf{R}^n] \subset \mathbb{KS}^2[\mathbf{R}^n]$, by theorem 12 $\mathfrak{F}$ extends to a bounded linear operator on $\mathbb{KS}^2[\mathbf{R}^n]$.

To prove that $\mathfrak{C}$ has a bounded extension, fix g in $\mathbf{L}^1[\mathbf{R}^n]$ and define $\mathfrak{C}_g$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$\mathfrak{C}_g(f)(\mathbf{x}) = \int g(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Once again, since $\mathfrak{C}_g$ is bounded on $\mathbf{L}^1[\mathbf{R}^n]$ and $\mathbf{L}^1[\mathbf{R}^n]$ is dense in $\mathbb{KS}^2[\mathbf{R}^n]$, by theorem 12 it extends to a bounded linear operator on $\mathbb{KS}^2[\mathbf{R}^n]$. Now use the fact that convolution is commutative to get that $\mathfrak{C}_f$ is a bounded linear operator on $\mathbf{L}^1[\mathbf{R}^n]$ for all $f \in \mathbb{KS}^2[\mathbf{R}^n]$. Another application of theorem 12 completes the proof. □

We now return to $\mathfrak{M}[\mathbf{R}^n]$.

Definition 14. *A uniformly bounded sequence* $\{\mu_k\} \subset \mathfrak{M}[\mathbf{R}^n]$ *is said to converge weakly to* μ ($\mu_n \xrightarrow{w} \mu$)*, if, for every bounded uniformly continuous function* $h(\mathbf{x})$*,*

$$\int_{\mathbf{R}^n} h(\mathbf{x})\,\mathrm{d}\mu_n \to \int_{\mathbf{R}^n} h(\mathbf{x})\,\mathrm{d}\mu.$$

Theorem 15. *If* $\mu_n \xrightarrow{w} \mu$ *in* $\mathfrak{M}[\mathbf{R}^n]$*, then* $\mu_n \xrightarrow{s} \mu$ *(strongly) in* $\mathbb{KS}^p[\mathbf{R}^n]$.

Proof. Since the characteristic function of a closed cube is a bounded uniformly continuous function, $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbf{R}^n]$ implies that

$$\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x})\,\mathrm{d}\mu_n \to \int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x})\,\mathrm{d}\mu$$

for each k, so that $\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathbf{KS}^p} = 0$. □

A little reflection gives

Theorem 16. *The space* $\mathbb{KS}^2[\mathbf{R}^n]$ *is a commutative Banach algebra with unit.*

In closing, it is clear that all bounded linear operators on $\mathbf{L}^p[\mathbf{R}^n]$ have extensions to $\mathbb{KS}^2[\mathbf{R}^n]$. It is easy to see that they also have densely defined closed extensions to $\mathbb{KS}^p[\mathbf{R}^n]$ for $p \neq 2$. We have not been able to show that these extensions are bounded.

4. Applications

4.1. Markov processes

In the study of Markov processes, two of the natural spaces on which to formulate the theory, $\mathbf{C}_b[\mathbb{R}^n]$, the space of bounded continuous functions, or $\mathbf{UBC}[\mathbb{R}^n]$, the bounded uniformly continuous functions, do not have the expected properties. It is well known that the semigroups associated with Markov processes, whose generators have unbounded coefficients, are not necessarily strongly continuous when defined on $\mathbf{C}_b[\mathbb{R}^n]$. This means that the generator of such a semigroup does not exist in the standard sense. As a consequence, a number of weaker (equivalent) definitions have been developed in the literature. For a good discussion of this and related problems see Lorenzi and Bertoldi [LB].

Definition 17. *A sequence of functions* $\{f_n\}$ *in* $\mathbf{C}_b[\mathbb{R}^n]$ *is said to converge to* f *in the mixed topology, written* $\tau^M\text{-}\lim f_n = f$*, if* $\sup_{n\in\mathbb{N}} \|f_n\|_\infty \leqslant M$ *and* $\|f_n - f\|_\infty \to 0$ *uniformly on every compact subset of* $\mathbb{R}^n$.

Theorem 18. *If* $\{f_n\}$ *converges to* f *in the mixed topology on* $\mathbf{C}_b[\mathbb{R}^n]$*, then* $\{f_n\}$ *converges to* f *in the norm topology of* $\mathbb{KS}^p[\mathbb{R}^n]$ *for each* $1 \leqslant p \leqslant \infty$.

Proof. It is easy to see that both $\mathbf{C}_b[\mathbb{R}^n]$ and $\mathbf{UBC}[\mathbb{R}^n]$ are subsets of $\mathbb{KS}^p[\mathbb{R}^n]$ for $1 \leqslant p \leqslant \infty$. Thus, it suffices to prove that τ^M-lim $f_n = f$ implies that $\lim_{n\to\infty} \|f_n - f\|_{\mathbb{KS}^p} = 0$. This now follows from the fact that each box, used in our definition of the $\mathbb{KS}^p[\mathbb{R}^n]$ norm, is a compact subset of $\mathbb{R}^n$. □

Theorem 19. *Suppose that $\hat{T}(t)$ is a transition semigroup defined on $\mathbf{C}_b[\mathbb{R}^n]$, with weak generator $\hat{A}$. Let $T(t)$ be the extension of $\hat{T}(t)$ to $\mathbb{KS}^p[\mathbb{R}^n]$. Then $T(t)$ is strongly continuous, and the extension A of $\hat{A}$ to $\mathbb{KS}^p[\mathbb{R}^n]$ is the strong generator of $T(t)$.*

Proof. First observe that the dual of $\mathbf{C}_b[\mathbb{R}^n]$ is $\mathfrak{M}[\mathbf{R}^n]$, which is contained in $\mathbb{KS}^p[\mathbb{R}^n]$ for $1 \leqslant p \leqslant \infty$. Thus we can apply theorem 12 to show that $\hat{T}(t)$ has a bounded extension to $\mathbb{KS}^2[\mathbb{R}^n]$. It is easy to show that the extended operator $T(t)$ is a semigroup. Since the τ^M topology on $\mathbf{C}_b[\mathbb{R}^n]$ is stronger than the norm topology on $\mathbb{KS}^2[\mathbb{R}^n]$, we see that the generator A of $T(t)$ is strong. □

4.2. Feynman path integral

Historically, the mathematics community has had two responses to the introduction of a new mathematical idea or method into physics. The first response has been to fit the idea or method into an existing framework. The second and more exciting is when such an idea or method leads to the development of a new branch of mathematics.

The most prevalent and successful response has been in finding an existing mathematical structure that will reasonably accommodate the physical theory and provide (at least) the framework for mathematical rigor. An excellent example of this is the introduction of matrix algebra into the Heisenberg formulation of quantum theory (e.g., matrix mechanics) by Born and Jordan [BHJ]. This made it possible for Schrödinger to show that, in the non-relativistic case, his wave mechanics was equivalent to Heisenberg's theory. This was later shown to be rigorously true mathematically via the unitary equivalence between $\mathbf{l}_2$ and $\mathbf{L}^2$ as separable Hilbert spaces (cf, von Neumann [VN2]). However, even in this case, we should not conclude that this is the complete story. There have always been physical advantages in looking at and working with some problems using the Heisenberg formulation. In fact, in 1964, Dirac strongly suggested on physical grounds that, at the quantum field level, Heisenberg's formulation is much more fundamental (see [BO], p 130). Furthermore, recent studies strongly indicate that the mathematical concept of isometric isomorphism is neither necessary nor sufficient for physical equivalence. (For example, it is known, [GZA], that the Dirac operator is non-local in time, while the square-root operator is non-local in space, but are unitarily equivalent.)

In some rare but important instances, there is no obvious mathematical structure which can completely accommodate the theory in the manner presented by physicists. In this case, mathematicians have extended and/or adapted an existing mathematical theory, developed new mathematical structures or suggested (in frustration) that any conclusions derived from the use of these ideas or methods are at least suspect. Over the last sixty years, all of the above positions have appeared in response to Feynman's introduction of his path integral into quantum theory.

Since his path integral is the object of this section, let us consider the simple case of a free particle in non-relativistic quantum theory in $\mathbb{R}^3$:

$$\mathrm{i}\hbar \frac{\partial \psi(\mathbf{x},t)}{\partial t} - \frac{\hbar^2}{2m}\Delta\psi(\mathbf{x},t) = 0, \qquad \psi(\mathbf{x},s) = \delta(\mathbf{x}-\mathbf{y}). \tag{4.1}$$

The solution can be computed directly:

$$\psi(\mathbf{x},t) = K[\mathbf{x},t;\mathbf{y},s] = \left[\frac{2\pi\mathrm{i}\hbar(t-s)}{m}\right]^{-3/2} \exp\left[\frac{\mathrm{i}m}{2\hbar}\frac{|\mathbf{x}-\mathbf{y}|^2}{(t-s)}\right].$$

Feynman wrote the above solution to equation (4.1) as

$$K[\mathbf{x},t;\mathbf{y},s]=\int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x}\mathcal{D}\mathbf{x}(\tau)\exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_s^t\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|^2\mathrm{d}\tau\right\},\tag{4.2}$$

where

$$\int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x}\mathcal{D}\mathbf{x}(\tau)\exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_s^t\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|^2\mathrm{d}\tau\right\}$$
$$=:\lim_{N\to\infty}\left[\frac{m}{2\pi\mathrm{i}\hbar\varepsilon(N)}\right]^{3N/2}\int_{\mathbf{R}^3}\prod_{j=1}^{N}\mathrm{d}\mathbf{x}_j\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\varepsilon(N)}(\mathbf{x}_j-\mathbf{x}_{j-1})^2\right]\right\},\tag{4.3}$$

with $\varepsilon(N)=(t-s)/N$.

Problems

Feynman's objective was to develop an approach to quantum theory which would avoid the use of a Hamiltonian. Equation (4.2) can be viewed as an attempt to 'apparently' define an integral over the space of all continuous paths of the exponential of an integral of the classical Lagrangian on configuration space. Thus, his objective was (partly) accomplished.

However, this approach (using the Lagrangian directly) has led to a new method for quantizing physical systems, called the path integral method. It is now used almost exclusively by large groups (in all branches of physics) and has also been used (formally) by researchers in both mathematics and mathematical physics. Thus, we must conclude that Feynman's formulation (as he proposed it) is both physically and mathematically distinct from those of Heisenberg and Schrödinger. (Feynman showed that it was equivalent to the other two.)

From a mathematical point of view, this leads to a number of problems:

- The kernel $K[\mathbf{x},t;\mathbf{y},s]$ and $\delta(\mathbf{x})$ are not in $L^2[\mathbf{R}^3]$, the standard space for quantum theory.
- The kernel $K[\mathbf{x},t;\mathbf{y},s]$ cannot be used to define a measure.

Thus, a natural question is: Does there exist a separable Hilbert space containing $\mathbf{K}[\mathbf{x},t;\mathbf{y},s]$ and $\delta(\mathbf{x})$ which also allows the convolution and Fourier transform as bounded operators? A positive answer to this question is necessary if we are to make sense of equation (4.3) and have a representation space for the Feynman formulation of quantum theory (as presented).

The properties of $\mathbb{KS}^2[\mathbf{R}^n]$ derived earlier suggest that it may be a more appropriate Hilbert space, compared to $\mathbf{L}^2[\mathbf{R}^n]$, for the Feynman formulation. It is easy to prove that both the position and momentum operators have closed densely defined extensions to $\mathbb{KS}^2[\mathbf{R}^n]$. Furthermore, the extensions of $\mathfrak{F}$ and $\mathfrak{C}$ insure that both the Schrödinger and Heisenberg theory have faithful representations on $\mathbb{KS}^2[\mathbf{R}^n]$.

Since $\mathbb{KS}^2[\mathbf{R}^n]$ contains the space of measures, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $\mathbb{KS}^2[\mathbf{R}^n]$ topology. (For example, $\left[\sin(\lambda\cdot\mathbf{x})/(\lambda\cdot\mathbf{x})\right]\in\mathbb{KS}^2[\mathbf{R}^n]$ and converges strongly to $\delta(\mathbf{x})$.) Thus, the finitely additive set function defined on the Borel sets (Feynman kernel [FH]): (with $m=1$ and $\hbar=1$)

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B]=\int_B(2\pi\mathrm{i}(t-s))^{-n/2}\exp\{\mathrm{i}|\mathbf{x}-\mathbf{y}|^2/2(t-s)\}\,\mathrm{d}\mathbf{y}$$

is in $\mathbb{KS}^2[\mathbf{R}^n]$ and $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B]\|_{\mathrm{KS}}\leqslant 1$, while $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B]\|_{\mathfrak{M}}=\infty$ (the total variation norm) and

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};s,B]=\int_{\mathbf{R}^n}\mathbb{K}_{\mathbf{f}}[t,\mathbf{x};\tau,\mathrm{d}\mathbf{z}]\mathbb{K}_{\mathbf{f}}[\tau,\mathbf{z};s,B],\quad\text{(HK-integral)}.$$

Definition 20. *Let* $\mathbf{P}_n = \{t_0, \tau_1, t_1, \tau_2, \ldots, \tau_n, t_n\}$ *be a HK-partition for a function* $\delta_n(s)$, $s \in [0, t]$ *for each n, with* $\lim_{n\to\infty} \Delta\mu_n = 0$ *(mesh). Set* $\Delta t_j = t_j - t_{j-1}$, $\tau_0 = 0$ *and, for* $\psi \in \mathbb{KS}^2[\mathbf{R}^n]$, *define*

$$\int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau); \mathbf{x}(0)] = \mathrm{e}^{-\lambda t} \sum_{k=0}^{[\![\lambda t]\!]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbf{R}^n} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(\tau_j); t_{j-1}, \mathrm{d}\mathbf{x}(\tau_{j-1})] \right\},$$

and

$$\int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(0)]\psi[\mathbf{x}(0)] = \lim_{\lambda\to\infty} \int_{\mathbf{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau); \mathbf{x}(0)]\psi[\mathbf{x}(0)] \tag{4.4}$$

whenever the limit exists.

Remark 21. In the above definition we have used the Poisson process. This is not accidental but appears naturally from a physical analysis of the information that is knowable in the micro-world (see [GZ]). In fact, it has been suggested by Kolokoltsov [KO] that such jump processes often provide another effective way to give meaning to the Feynman path integral and also offers a nice approach to Feynman diagrams.

The next result is now elementary, since $\mathbb{KS}^2[\mathbf{R}^n]$ is closed under convolution.

Theorem 22. *The function* $\psi(\mathbf{x}) \equiv 1 \in \mathbb{KS}^2[\mathbf{R}^n]$ *and*

$$\int_{\mathbf{R}^{n[s,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(s)] = \mathbb{K}_{\mathbf{f}}[t, \mathbf{x}; s, \mathbf{y}] = \frac{1}{\sqrt{[2\pi \mathrm{i}(t-s)]^n}} \exp\{\mathrm{i}|\mathbf{x}-\mathbf{y}|^2/2(t-s)\}.$$

The above result is what Feynman was trying to obtain without the appropriate space. A more general (sum over paths) result, that covers almost all application areas, will appear later, where these spaces have been used to provide a generalization of the constructive representation theory for the Feynman operator calculus (see [GZ1] and also [GZ] for other applications).

If we treat $K[\mathbf{x}, t; \mathbf{y}, s]$ as the kernel for an operator acting on good initial data, then a partial solution has been obtained by a number of workers. (See [GZ1] for references to all the important contributions in this direction.)

A related approach to the Feynman path integral can be found in the work of Fujiwara and Kumano-go (see [FK1], [FK2] and references therein). For a survey of this approach, see [FK3]. They have systematically developed a time-slicing approximation method that covers a large portion of classical quantum theory. They restrict themselves to scalar potentials with polynomial growth. However, their method seems general enough to eventually include the additional cases. (They show the power of their approach by providing an analytic formula for the second term of the semi-classical asymptotic expansion of the Feynman path integral.)

4.3. Examples

A standard method is to compute the Wiener path integral for the problem under consideration and then use analytic continuation in the mass to provide a rigorous meaning for the Feynman path integral. The following example provides a path integral representation for a problem that cannot be solved using analytic continuation via a Gaussian kernel (see Gill and Zachary [GZ3]). It is shown that, if the vector potential $\mathbf{A}$ is constant, $\mu = mc/\hbar$, and $\boldsymbol{\beta}$ is the standard beta matrix, then the solution to the square-root equation for a spin 1/2 particle:

$$\mathrm{i}\hbar\partial\psi(\mathbf{x}, t)\big/\partial t = \left\{\beta\sqrt{c^2\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4}\right\}\psi(\mathbf{x}, t),\ \psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}),$$

is given by

$$\psi(\mathbf{x}, t) = \mathbf{U}[t, 0]\psi_0(\mathbf{x}) = \int_{\mathbf{R}^3} \exp\left\{\frac{\mathrm{i}e}{2\hbar c}(\mathbf{x}-\mathbf{y})\cdot\mathbf{A}\right\}\mathbf{K}[\mathbf{x}, t; \mathbf{y}, 0]\,\psi_0(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

where

$$\mathbf{K}[\mathbf{x}, t; \mathbf{y}, 0] = \frac{\mathrm{i}ct\mu^2\beta}{4\pi}\begin{cases} \dfrac{-H_2^{(1)}[\mu(c^2t^2 - ||\mathbf{x}-\mathbf{y}||^2)^{1/2}]}{[c^2t^2 - ||\mathbf{x}-\mathbf{y}||^2]}, & ct < -||\mathbf{x}||, \\ \dfrac{-2\mathrm{i}K_2[\mu(||\mathbf{x}-\mathbf{y}||^2 - c^2t^2)^{1/2}]}{\pi[||\mathbf{x}-\mathbf{y}||^2 - c^2t^2]}, & c|t| < ||\mathbf{x}||, \\ \dfrac{H_2^{(2)}[\mu(c^2t^2 - ||\mathbf{x}-\mathbf{y}||^2)^{1/2}]}{[c^2t^2 - ||\mathbf{x}-\mathbf{y}||^2]}, & ct > ||\mathbf{x}||. \end{cases}$$

The function $K_2(\cdot)$ is a modified Bessel function of the third kind of second order, while $H_2^{(1)}, H_2^{(2)}$ are Hankel functions (see Gradshteyn and Ryzhik [GRRZ]). Thus, we have a kernel that is far from the standard form. This example was first introduced in [GZ2], where we only considered the kernel for the Bessel function term. In that case, it was shown that, under appropriate conditions, that term will reduce to the free-particle Feynman kernel and, if we set $\mu = 0$, we get the kernel for a (spin 1/2) massless particle. In closing this section, we remark that the square-root operator is unitarily equivalent to the Dirac operator (in the case discussed).

4.4. The kernel problem

Since any semigroup that has a kernel representation will automatically generate a path integral via the reproducing property, a fundamental question is Under what general conditions can we expect a given (time-independent) generator of a semigroup to have an associated kernel? In this section we discuss a class of general conditions for unitary groups. It will be clear that the results of this section carry over to semigroups with minor changes.

Let $A(\mathbf{x}, \mathbf{p})$ denote a $k \times k$ matrix operator $[A_{ij}(\mathbf{x}, \mathbf{p})], i, j = 1, 2, \ldots, k$, whose components are pseudodifferential operators with symbols $a_{ij}(\mathbf{x}, \boldsymbol{\eta}) \in \mathbb{C}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and we have, for any multi-indices α and β,

$$\left|a_{ij(\beta)}^{(\alpha)}(\mathbf{x}, \boldsymbol{\eta})\right| \leqslant C_{\alpha\beta}(1 + |\boldsymbol{\eta}|)^{m-\xi|\alpha|+\delta|\beta|}, \tag{4.5}$$

where

$$a_{ij(\beta)}^{(\alpha)}(\mathbf{x}, \boldsymbol{\eta}) = \partial^\alpha \mathbf{p}^\beta a_{ij}(\mathbf{x}, \boldsymbol{\eta}),$$

with $\partial_l = \partial/\partial\eta_l$, and $p_l = (1/\mathrm{i})(\partial/\partial x_l)$. The multi-indices are defined in the usual manner by $\alpha = (\alpha_1, \ldots, \alpha_n)$ for integers $\alpha_j \geqslant 0$, and $|\alpha| = \sum_{j=1}^n \alpha_j$, with similar definitions for β. The notation for derivatives is $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ and $\mathbf{p}^\beta = p_1^{\beta_1}\cdots p_n^{\beta_n}$. Here, m, β and δ are real numbers satisfying $0 \leqslant \delta < \xi$. Equation (4.5) states that each $a_{ij}(\mathbf{x}, \boldsymbol{\eta})$ belongs to the symbol class $S_{\xi,\delta}^m$ (see [SH]).

Let $a(\mathbf{x}, \boldsymbol{\eta}) = [a_{ij}(\mathbf{x}, \boldsymbol{\eta})]$ be the matrix-valued symbol for $A(\mathbf{x}, \boldsymbol{\eta})$, and let $\lambda_1(\mathbf{x}, \boldsymbol{\eta})\cdots\lambda_k(\mathbf{x}, \boldsymbol{\eta})$ be its eigenvalues. If $|\cdot|$ is the norm in the space of $k \times k$ matrices, we assume that the following conditions are satisfied by $a(\mathbf{x}, \boldsymbol{\eta})$. For $0 < c_0 < |\boldsymbol{\eta}|$ and $\mathbf{x} \in \mathbb{R}^n$ we have

(i) $\left|a_{(\beta)}^{(\alpha)}(\mathbf{x}, \boldsymbol{\eta})\right| \leqslant C_{\alpha\beta}\,|a(\mathbf{x}, \boldsymbol{\eta})|\,(1 + |\boldsymbol{\eta}|)^{-\xi|\alpha|+\delta|\beta|}$ (hypoellipticity),

(ii) $\lambda_0(\mathbf{x}, \boldsymbol{\eta}) = \max_{1\leqslant j\leqslant k} \mathrm{Re}\lambda_j(\mathbf{x}, \boldsymbol{\eta}) < 0$,

(iii) $\frac{|a(\mathbf{x},\boldsymbol{\eta})|}{|\lambda_0(\mathbf{x},\boldsymbol{\eta})|} = O[(1+|\boldsymbol{\eta}|)^{(\xi-\delta)/(2k-\varepsilon)}]$, $\varepsilon > 0$.

We assume that $A(\mathbf{x}, \mathbf{p})$ is a selfadjoint generator of a unitary group $U(t, 0)$, so that

$$U(t,0)\psi_0(\mathbf{x}) = \exp[(\mathrm{i}/\hbar)tA(\mathbf{x},\mathbf{p})]\psi_0(\mathbf{x}) = \psi(\mathbf{x},t)$$

solves the Cauchy problem

$$(\mathrm{i}\hbar)\partial\psi(\mathbf{x},t)/\partial t = A(\mathbf{x},\mathbf{p})\psi(\mathbf{x},t), \qquad \psi(\mathbf{x},t) = \psi_0(\mathbf{x}). \tag{4.6}$$

Definition 23. *We say that $Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)$ is a symbol for the Cauchy problem (4.6) if $\psi(\mathbf{x}, t)$ has a representation of the form*

$$\psi(\mathbf{x},t) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\mathbf{x},\boldsymbol{\eta})} Q(\mathbf{x},t,\boldsymbol{\eta},0)\hat{\psi}_0(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}. \tag{4.7}$$

It is sufficient that ψ_0 belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, which is contained in the domain of $A(\mathbf{x}, \mathbf{p})$, in order that (4.7) makes sense.

Following Shishmarev [SH], and using the theory of Fourier integral operators, we can define an operator-valued kernel for $U(t, 0)$ by

$$K(\mathbf{x},t;\mathbf{y},0) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\mathbf{x}-\mathbf{y},\boldsymbol{\eta})} Q(\mathbf{x},t,\boldsymbol{\eta},0)\,\mathrm{d}\boldsymbol{\eta},$$

so that

$$\psi(\mathbf{x},t) = U(t,0)\psi_0(\mathbf{x}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} K(\mathbf{x},t;\mathbf{y},0)\psi_0(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{4.8}$$

The following results are due to Shishmarev [SH].

Theorem 24. *If $A(\mathbf{x}, \mathbf{p})$ is a selfadjoint generator of a strongly continuous unitary group with domain D, $\mathcal{S}(\mathbb{R}^n) \subset D$ in $L^2(\mathbb{R}^n)$, such that conditions (1)–(3) are satisfied. Then there exists precisely one symbol $Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)$ for the Cauchy problem (4.6).*

Theorem 25. *If we replace our condition (3) in theorem 24 by the stronger condition*

$$(3') \quad \frac{|a(\mathbf{x},\boldsymbol{\eta})|}{|\lambda_0(\mathbf{x},\boldsymbol{\eta})|} = O\left[(1+|\boldsymbol{\eta}|)^{(\xi-\delta)/(3k-1-\varepsilon)}\right], \qquad \varepsilon > 0, |\boldsymbol{\eta}| > c_0,$$

then the symbol $Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)$ of the Cauchy problem (4.6) has the asymptotic behavior near $t = 0$:

$$Q(\mathbf{x},t,\boldsymbol{\eta},0) = \exp[-(\mathrm{i}/\hbar)ta(\mathbf{x},\boldsymbol{\eta})] + o(1),$$

uniformly for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Now, using theorem 24 we see that, under the stronger condition (3'), the kernel $K(\mathbf{x}, t; \mathbf{y}, 0)$ satisfies

$$K(\mathbf{x},t;\mathbf{y},0) = \int_{\mathbb{R}^n} \exp[\mathrm{i}(\mathbf{x}-\mathbf{y},\boldsymbol{\eta}) - (\mathrm{i}/\hbar)ta(\mathbf{x},\boldsymbol{\eta})]\frac{\mathrm{d}\boldsymbol{\eta}}{(2\pi)^{n/2}} + \int_{\mathbb{R}^n} \exp[\mathrm{i}(\mathbf{x}-\mathbf{y},\boldsymbol{\eta})]\frac{\mathrm{d}\boldsymbol{\eta}}{(2\pi)^{n/2}}o(1).$$

In order to see the power of $\mathbb{KS}^2(\mathbf{R}^n)$, first note that $A(\mathbf{x}, \mathbf{p})$ has a selfadjoint extension to $\mathbb{KS}^2(\mathbf{R}^n)$, which also generates a unitary group. This means that we can construct a path integral in the same (identical) way as was done for the free-particle propagator (i.e., for all Hamiltonians with symbols in $\mathcal{S}^m_{\alpha,\delta}$). Furthermore, it follows that the same comment applies to any Hamiltonian that has a kernel representation, independent of its symbol class. The important point of this discussion is that no initial data nor Gaussian form for the kernel is required!

4.5. Discussion

A natural reaction to any suggestion that we replace the Lebesgue integral by one based on a finitely additive measure would be negative. After all we would lose all of the advantages of the powerful theorems (dominated convergence theorem, monotone convergence theorem, etc) that depend on the countable additivity of the measure. Those strongly vested in using $\mathbf{L}^2$ for the C^*-algebra approach to quantum theory via the GNS construction may also feel obliged to object to such a proposed change. These are all reasonable concerns. However, we do not lose any of the powerful theorems found via countable additivity. First of all the HK-measure is an extension of the Lebesgue measure so that all of its power is still available to us. In fact, Henstock has extended each of the standard theorems to the HK-integral (see [HS]). Those concerned with the C^*-algebra approach to quantum theory need not be concerned since $\mathbb{KS}^2$ is a separable Hilbert space and is also amenable to the GNS construction.

5. Conclusion

In this paper we have shown how to construct a natural class of separable Banach spaces $\mathbb{KS}^p$ which parallels the standard L^p spaces but contains them as dense compact embeddings. These spaces are of particular interest because they contain the Henstock–Kurzweil integrable functions and the HK-measure, which generalizes the Lebesgue measure. We have also constructed the corresponding spaces $\mathbb{KS}^m_p$ of Sobolev type.

We have used $\mathbb{KS}^2$ to construct the free-particle path integral in the manner originally intended by Feynman. We have suggested that $\mathbb{KS}^2$ has a claim as the natural representation space for the Feynman formulation of quantum theory in that it allows representations for both the Heisenberg and Schrödinger representations, a property not shared by $\mathbf{L}^2$.

In the analytical theory of Markov processes, it is well known that, in general, the semigroup $T(t)$ associated with the process is not strongly continuous on $\mathbf{C}_b[\mathbb{R}^n]$, the space of bounded continuous functions or on $\mathbf{UBC}[\mathbb{R}^n]$, the bounded uniformly continuous functions. We have shown that the weak generator defined by the mixed locally convex topology on $\mathbf{C}_b[\mathbb{R}^n]$ is a strong generator on $\mathbb{KS}^p[\mathbb{R}^n]$ (e.g., $T(t)$ is strongly continuous on $\mathbb{KS}^p[\mathbb{R}^n]$ for $1 \leqslant p \leqslant \infty$).

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